WEIGHTED WEAK TYPE ESTIMATES FOR SQUARE FUNCTIONS

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ABSTRACT. For $1 < p < \infty$ and weight $w \in A_p$, the following weak-type inequality holds for a Littlewood-Paley square function S,

$$\|Sf\|_{L^{p,\infty}(w)} \lesssim [w]_{A_p}^{\max\{\frac{1}{2},\frac{1}{p}\}} \phi([w]_{A_p}) \|f\|_{L^p(w)} .$$

where $\phi_p(x) = 1$ for $1 < p < 2$ and $\phi_p(x) = 1 + \log x$ for $2 \leq p$. Up to the logarithmic term, these estimates are sharp.

1. Introduction

Our focus is on weak-type estimates for square functions on weighted L^p spaces, for Muckenhoupt A_p weights. Following M. Wilson [16] define the intrinsic square function G_α as follows.

1.1. **Definition.** Let C_α be the collection of functions γ supported in the unit ball with mean zero and such that $|\gamma(x) - \gamma(y)| \leq |x - y|^\alpha$. For $f \in L^1_{loc}(\mathbb{R}^n)$ let

$$A_\alpha f(x,t) = \sup_{\gamma \in C_\alpha} |f * \gamma_t(x)|$$

where $\gamma_t(x) = t^{-n}\gamma(xt^{-n})$ and take

$$G_\alpha f(x) = \Big(\int_{\Gamma(x)} A_\alpha f(y,t)^2 \frac{dy\,dt}{t^{n+1}}\Big)^{\frac{1}{2}}$$

where $\Gamma(x) := \{(y,t) \in \mathbb{R}^{n+1}_+ \;:\; |y| < t\}$ is the cone of aperture one in the upper-half plane.

This square function dominates many other square functions. Recall the definition of A_p weights.

1.2. **Definition.** Let $1 < p < \infty$. A weight w is in A_p if w has density $w(x)$, we have $w(x) > 0$ a.e., and for $\sigma(x) := w(x)^{1-\frac{p}{p-1}}$ there holds

$$[w]_{A_p} := \sup_Q \frac{w(Q)}{|Q|}\Big[\frac{\sigma(Q)}{|Q|}\Big]^{p-1} < \infty$$

where the supremum is formed over all cubes $Q \subset \mathbb{R}^n$.

The main result of this note is as follows.

Research supported in part by grant NSF-DMS 0968499, and a grant from the Simons Foundation (#229596 to Michael Lacey). The first author benefited from the research program Operator Related Function Theory and Time-Frequency Analysis at the Centre for Advanced Study at the Norwegian Academy of Science and Letters in Oslo during 2012ï£¡2013.

1.3. **Theorem.** *For $1 < p < 3$, $0 < \alpha \le 1$, and $w \in A_p$ the following inequality holds.*

$$\|G_\alpha f\|_{L^p(w)\to L^{p,\infty}(w)} \lesssim [w]_{A_p}^{\max\{\frac{1}{2},\frac{1}{p}\}} \phi([w]_{A_p}) \|f\|_{L^p(w)}\,, \tag{1.4}$$

$$\text{where}\quad \phi([w]_{A_p}) := \begin{cases} 1 & 1 < p < 2 \\ (1+\log[w]_{A_p}) & 2 \le p < 3 \end{cases}$$

By example, we will show that the power on $[w]_{A_p}$, but not the logarithmic term, is sharp. This result can be contrasted with these known results. First, for the maximal function M, one has the familiar estimate of Buckley [1],

$$\|M\|_{L^p(w)\to L^{p,\infty}(w)} \lesssim [w]_{A_p}^{1/p}\,, \qquad 1 < p < \infty\,.$$

Thus, the square function estimate equals that for M for $1 < p < 2$, but is otherwise larger. There is also the recent sharp estimate of the strong type norm of G_α:

$$\|G_\alpha\|_{L^p(w)\to L^p(w)} \lesssim [w]_{A_p}^{\max\{\frac{1}{2},\frac{1}{p-1}\}}.$$

The weak-type estimate above is smaller for all values of $1 < p < 3$, and is otherwise larger by the logarithmic term. The case of $p = 2$ in the dyadic strong-type inequality was proved by Wittwer [18], also see [4]. The dyadic case, for general p, was proved by Cruz-Uribe-Martell-Perez [3], while the inequality as above is the main result of Lerner's paper [9].

The case $p = 1$ of Theorem 1.3 holds more generally. Chanillo-Wheeden [2], first for the area function, and Wilson [16, 17], showed that for any weight w,

$$w\{G_\alpha f > \lambda\} \lesssim \frac{1}{\lambda}\int_{\mathbb{R}^n} |f| \cdot Mw\, dx \tag{1.5}$$

where M is the Hardy-Littlewood maximal function. In particular, (1.4) holds for $p = 1$.

There are interesting points of comparison with the weak-type estimates for Calderón–Zygmund operators. Hytönen [5] established the strong type estimate. For T an $L^2(\mathbb{R}^n)$ bounded Calderon-Zygmund operator, there holds

$$\|T\|_{L^p(w)\to L^p(w)} \lesssim [w]_{A_p}^{\max\{\frac{1}{p-1},1\}} \qquad 1 < p < \infty\,.$$

Hytönen et. al. [7] the weak-type estimate

$$\|T\|_{L^p(w)\to L^{p,\infty}(w)} \lesssim [w]_{A_p}, \qquad 1 < p < \infty\,.$$

But, the L^1-endpoint variant of (1.5) fails, as was shown by Reguera [12], for the dyadic case and Reguera-Thiele [13] for the continuous case. Specializing to the case where $w \in A_1$, Lerner-Ombrosi-Perez [10] have shown that

$$\|T\|_{L^1(w)\to L^{1,\infty}(w)} \lesssim [w]_{A_1}(1+\log[w]_{A_1}).$$

And, in a very interesting twist, some power of the logarithm is necessary, by the argument of Nazarov et al. [11]. It seems entirely plausible to us that in the case of $p = 2$ in (1.4), that some power of the logarithm is required.

Acknowledgment. The question of looking at the weak-type inequalities was suggested to us by A. Volberg [15], and we thank S. Petermichl for conversations about the problem.

2. Proof of Theorem 1.3

Our argument will apply the Lerner median inequality [8]—a widely used technique, see [3,6,9], among other papers. To this end, we need some definitions. For a constant $\rho > 0$, let us set ρQ to be the cube with the same center as Q, and side length $|\rho Q|^{1/n} = \rho|Q|^{1/n}$. For any cube Q, set $|Q|\langle f\rangle_Q := \int_Q f \, dx$. We say that a collection of dyadic cubes $\mathcal{S}$ is *sparse* if there holds

$$\Bigl|\bigcup\{Q' \in \mathcal{S} \;:\; Q' \subsetneq Q\}\Bigr| < \tfrac{1}{2}|Q| \,, \qquad Q \in \mathcal{S} \,.$$

For a sparse collection of cubes $\mathcal{S}$ and $\rho > 1$ we define

$$\Bigl(T_{\mathcal{S},\rho} f\Bigr)^2 := \sum_{Q\in\mathcal{S}} \langle f\rangle_{\rho Q}^2 \mathbf{1}_Q \,.$$

Fix f supported on a dyadic cube Q_0. By application to Lerner's median inequality (compare to [9, (5.8)]), for $N \to \infty$, there are constants $m_N \to 0$ so that there is a sparse collection of cubes $\mathcal{S}_N$ contained in NQ_0 so that, for $\rho = 45$, the following pointwise estimate holds.

$$|G_\alpha f(x)^2 - m_N| \cdot \mathbf{1}_{NQ_0}(x) \lesssim Mf(x)^2 + T_{\mathcal{S}_N} f(x)^2 \,.$$

Therefore, in order to estimate the $L^{p,\infty}(w)$ norm of $G_\alpha f$, it suffices to estimate $L^p(w)$ norm of Mf and of $T_{\mathcal{S}} f$, for any sparse collection of cubes $\mathcal{S}$.

Now, by Buckley's bounds [1], $\|M\|_{L^p \to L^{p,\infty}(w)} \lesssim [w]_{A_p}^{1/p}$. As a result, to obtain Theorem 1.3 it suffices to show this Theorem.

2.1. **Theorem.** *For $1 < p < 3$, weight $w \in A_p$, any sparse collection of cubes $\mathcal{S}$ and any $\rho \geq 1$, there holds*

$$\|T_{\mathcal{S}}\|_{L^p(w) \to L^{p,\infty}(w)} \lesssim [w]_{A_p} \phi([w]_{A_p}) \,.$$

We turn to the proof of this estimate. With $\rho > 1$ fixed, it is clear that it suffices to consider collections $\mathcal{S}$ which satisfy this strengthening of the defintion of sparseness: On the one hand,

$$\Bigl|\bigcup\{Q' \in \mathcal{S} \;:\; Q' \subsetneq Q\}\Bigr| < \tfrac{|Q|}{8\rho^n} \,, \qquad Q \in \mathcal{S} \,.$$

and on the other, if $Q \neq Q' \in \mathcal{S}$ and $|Q| = |Q|$, then $\rho Q \cap \rho Q' = \emptyset$. We can assume these conditions, as a sparse collection is the union of $O(\rho^{n+1})$ subcollections which meet these conditions, and we are not concerned with the effectiveness of our estimates in ρ.

Let $\mathcal{S}^1$ consist of all Q such that $\langle f\rangle_{\rho Q} > 1$. Then if $Q \in \mathcal{S}^1$ we have $Q \subset \{Mf > 1\}$ so that

$$w\Bigl\{\sum_{Q\in\mathcal{S}^1} \langle f\rangle_{\rho Q}^2 \mathbf{1}_Q > 1\Bigr\} \leq w\Bigl\{\bigcup_{Q\in\mathcal{S}^1} Q\Bigr\} \leq w\{Mf > 1\} \lesssim [w]_{A_p} \|f\|_{L^p(w)}^p.$$

We split the remaining cubes into disjoint collections setting

$$\mathcal{S}_\ell := \{Q \in \mathcal{S} \;:\; 2^{-\ell-1} < \langle f\rangle_{\rho Q} \leq 2^{-\ell}\}, \qquad \ell = 0, 1, \ldots,$$

Now let $E(Q) = \rho Q \setminus R(Q)$ where and $R(Q) = \bigcup\{\rho Q' : Q' \subsetneq \rho Q,\ Q' \in \mathcal{S}_\ell\}$. Notice, that $|R(Q)| < \frac{1}{8}|\rho Q|$, whence

$$\begin{aligned}\langle f \mathbf{1}_{E(Q)} \rangle_{\rho Q} &= \langle f \rangle_{\rho Q} - \langle f \mathbf{1}_{R(Q)} \rangle_{\rho Q} \\ &\geq \langle f \rangle_{\rho Q} - 8^{-1} \langle f \rangle_{R(Q)} \\ &\geq 2^{-\ell-1} - 8^{-1} 2^{-\ell} \gtrsim 2^{-\ell}\,.\end{aligned}$$

That is, we have good lower bound on these averages, and moreover the sets $E(Q)$ are pairwise disjoint in $Q \in \mathcal{S}_\ell$. We will estimate

$$\sum_{Q \in \mathcal{S}_\ell} 2^{-2\ell} \mathbf{1}_Q \lesssim \sum_{Q \in \mathcal{S}_\ell} \langle f \mathbf{1}_{E(Q)} \rangle_{\rho Q}^2 \mathbf{1}_Q\,. \tag{2.2}$$

The following lemma is elementary.

2.3. **Lemma.** *Let $\mathcal{T}$ be a collection of cubes. We have, for $1 < p < \infty$, and sequences $\{g_Q : Q \in \mathcal{T}\}$ of non-negative functions,*

$$\Big\| \Big[\sum_{Q \in \mathcal{T}} \langle g_Q \rangle_{\rho Q}^p \mathbf{1}_Q \Big]^{1/p} \Big\|_{L^p(w)} \lesssim [w]_{A_p}^{1/p} \Big\| \Big[\sum_{Q \in \mathcal{T}} g_Q^p \Big]^{1/p} \Big\|_{L^p(w)}$$

Proof. This is a well-known estimate on the A_p norm of simple averaging operators. Writing $\sigma(x) = w(x)^{1-p'}$, we will exchange out an average over Lebesgue measure for an average over σ-measure. Thus, set

$$\langle \psi \rangle_Q^\sigma := \sigma(Q)^{-1} \int_Q \psi \; d\sigma\,.$$

We can estimate as follows.

$$\begin{aligned}\int_{\mathbb{R}^n} \langle g \rangle_{\rho Q}^p \mathbf{1} \; dw &= \Big(\langle g\sigma^{-1} \rangle_{\rho Q}^\sigma \Big)^p \Big(\frac{\sigma(\rho Q)}{|\rho Q|} \Big)^p w(\rho Q) \\ &\leq [w]_{A_p} \sigma(Q) \Big(\langle g\sigma^{-1} \rangle_{\rho Q}^\sigma \Big)^p \\ &= [w]_{A_p} \int_{\mathbb{R}^n} g^p \sigma^{-p} \; d\sigma = [w]_{A_p} \int_{\mathbb{R}^n} g^p \; dw.\end{aligned}$$

And the Lemma is a trivial extension of this inequality. □

2.1. **The Case of $1 < p < 2$.** We let $k_\epsilon \simeq \epsilon^{-1}$ be a constant such that

$$w\Big\{ \sum_{Q \in \mathcal{S} \setminus \mathcal{S}^1} \langle f \rangle_{\rho Q}^2 \mathbf{1}_Q > k_\epsilon \Big\} = w\Big\{ \sum_{\ell=0}^{\infty} \sum_{Q \in \mathcal{S}_\ell} 2^{-2\ell} \mathbf{1}_Q > \sum_{\ell=0}^{\infty} 2^{-\epsilon\ell} \Big\} \;\; \leq \sum_{\ell=0}^{\infty} w\Big\{ \sum_{Q \in \mathcal{S}_\ell} 2^{-2\ell} \mathbf{1}_Q > 2^{-\epsilon\ell} \Big\}.$$

For fixed ℓ we may estimate, using (2.2), and exchanging out a square for a pth power,

$$\begin{aligned}w\Big\{ \sum_{Q \in \mathcal{S}_\ell} 2^{-2\ell} \mathbf{1}_Q > 2^{-\epsilon\ell} \Big\} &\leq w\Big\{ \sum_{Q \in \mathcal{S}_\ell} \langle f \mathbf{1}_{E(Q)} \rangle_{\rho Q}^p \mathbf{1}_Q \gtrsim 2^{(2-p-\epsilon)\ell} \Big\} \\ &\lesssim [w]_{A_p} 2^{-(2-p-\epsilon)p/2\ell} \|f\|_{L^p(w)}^p\end{aligned}$$

where in the last inequality we have used Lemma 2.3. Choosing $\epsilon = 1 - p/2$ and summing over ℓ gives the result.

2.2. **The case of $p = 2$.** Of course the estimate is a bit crude. For a large constant C, take ℓ_0 to be the integer part of $C(1 + \log_2 [w]_{A_2})$. Estimate

$$\begin{aligned} w\Big\{ \sum_{Q \in \mathcal{S} \setminus \mathcal{S}^1} \langle f \rangle_{\rho Q}^2 \mathbf{1}_Q > 2 \Big\} &\le w\Big\{ \sum_{\ell=0}^{\ell_0 - 1} \sum_{Q \in \mathcal{S}_\ell} \langle f \rangle_{\rho Q}^2 \mathbf{1}_Q > 1 \Big\} + w\Big\{ \sum_{\ell=\ell_0}^{\infty} \sum_{Q \in \mathcal{S}_\ell} \langle f \rangle_{\rho Q}^2 \mathbf{1}_Q > \sum_{\ell=\ell_0}^{\infty} 2^{-\ell/8} \Big\} \\ &\le \sum_{\ell=0}^{\ell_0-1} w\Big\{ \sum_{Q \in \mathcal{S}_\ell} \langle f \rangle_{\rho Q}^2 \mathbf{1}_Q > \frac{1}{\ell_0} \Big\} + \sum_{\ell=\ell_0}^{\infty} w\Big\{ \sum_{Q \in \mathcal{S}_\ell} \langle f \rangle_{\rho Q}^2 \mathbf{1}_Q \gtrsim 2^{-\ell/8} \Big\} \end{aligned}$$

Recall the the A_∞ property for A_2 weights: For any cube Q and $E \subset Q$, with $|E| < \frac{1}{2}|Q|$, there holds

$$w(E) < \Big(1 - \tfrac{c}{[w]_{A_2}}\Big) w(Q)\,,$$

for absolute choice of constant c. Applying this in an inductive fashion, we see that

$$\begin{aligned} w\{ \sum_{Q \in \mathcal{S}_\ell} \langle f \rangle_{\rho Q}^2 \mathbf{1}_Q \gtrsim 2^{-\ell/8} \} &\le w\{ \sum_{Q \in \mathcal{S}_\ell} \mathbf{1}_Q \gtrsim 2^{15\ell/8} \} \\ &\lesssim \exp((-c 2^{15\ell/8})/[w]_{A_2}) w\{ \bigcup \{ Q \,:\, Q \in \mathcal{S}_\ell \} \} \\ &\lesssim [w]_{A_2} 2^{\ell} \exp(-c 2^{15\ell/8}/[w]_{A_2}) \|f\|_{L^2(w)}^2 \,. \end{aligned}$$

where $0 < c < 1$ is a fixed constant. This is summable in $\ell \ge \ell_0$ to at most a constant, for C sufficiently large.

For the case of $0 \le \ell < \ell_0$, we use the estimate of Lemma 2.3 to obtain

$$\sum_{\ell=0}^{\ell_0-1} w\{ \sum_{Q \in \mathcal{S}_\ell} \langle f \mathbf{1}_{E(Q)} \rangle_{\rho Q}^2 \mathbf{1}_Q > \frac{1}{\ell_0} \} \lesssim \ell_0^2 [w]_{A_2} \|f\|_{L^2(w)}^2 = [w]_{A_2} (1 + \log [w]_{A_2})^2 \|f\|_{L^2(w)}^2$$

concluding the proof of this case.

2.3. **The case of $2 < p$.** The case of $p = 2$ is the critical case, and so the case of p larger than 2 follows from extrapolation. However, here we are extrapolating weak-type estimates. It is known that this is possible, with estimates on constants. We outline the familiar argument as found in [10].

We have

$$\begin{aligned} w\Big\{ \Big[\sum_{Q \in \mathcal{S}} \langle f \rangle_Q^2 \mathbf{1}_Q \Big]^{1/2} > 1 \Big\}^{\frac{1}{p}} &= \Big(w\Big\{ \Big[\sum_{Q \in \mathcal{S}} \langle f \rangle_Q^2 \mathbf{1}_Q \Big]^{1/2} > 1 \Big\}^{\frac{2}{p}} \Big)^{\frac{1}{2}} \\ &= \Big(hw\Big\{ \Big[\sum_{Q \in \mathcal{S}} \langle f \rangle_Q^2 \mathbf{1}_Q \Big]^{1/2} > 1 \Big\} \Big)^{\frac{1}{2}} \end{aligned}$$

for $h \in L^{q'}(w)$ with norm 1, where $q = \frac{p}{2}$. Now by the Rubio de Francia algorithm there is a function H such that

i. $h \le H$
ii. $\|H\|_{L^{q'}(w)} \lesssim \|h\|_{L^{q'}(w)}$
iii. $Hw \in A_1$
iv. $[Hw]_{A_1} \lesssim [w]_{A_p}$.

We can continue,

$$\begin{aligned}\Big(hw\Big\{\Big[\sum_{Q\in\mathcal{S}}\langle f\rangle_Q^2\mathbf{1}_Q\Big]^{1/2} > 1\Big\}\Big)^{\frac12} &\le \Big(Hw\Big\{\Big[\sum_{Q\in\mathcal{S}}\langle f\rangle_Q^2\mathbf{1}_Q\Big]^{1/2} > 1\Big\}\Big)^{\frac12}\\ &\lesssim \left([Hw]_{A_2}\Big(1+\log{[Hw]_{A_2}}\Big)^2\int_{\mathbb{R}} f^2 Hw\right)^{\frac12}\\ &\lesssim [Hw]_{A_2}^{\frac12}\Big(1+\log{[Hw]_{A_2}}\Big)\|f\|_{L^p(w)}\|H\|_{L^{q'}(w)}\\ &\lesssim [w]_{A_p}^{\frac12}\Big(1+\log{[w]_{A_p}}\Big)\|f\|_{L^p(w)}.\end{aligned}$$

2.4. *Remark.* The estimate in Theorem 2.1 is a weighted estimate for a vector-valued dyadic positive operator. One of us [14] has characterized such inequalities in terms of testing conditions. Using this condition, we could not succeed in eliminating the logarithmic estimate in the case of $p = 2$. It did however suggest one of the examples in the next section.

3. Examples

The usual example of a power weight in one dimension $w(x) = |x|^{\epsilon-1}$, with $0 < \epsilon < 1$ has $[w]_{A_p} \simeq \epsilon^{-1}$. It is straight forward to see that for $\sigma(x) = w(x)^{1-p'}$, and appropriate constant $c = c(p)$, we have

$$c^p w\{S\mathbf{1}_{[0,1]} > c\} = c^p w([0,1]) \simeq \epsilon^{-1}$$

whereas $\|\mathbf{1}_{[0,1]}\|_{L^p(\sigma)} \simeq 1$. Hence, the smallest power on $[w]_{A_p}$ that we can have is $\frac{1}{p}$.

There is a finer example, expressed through the dual inequality, which shows that the power on $[w]_{A_p}$ can never be less than $\frac12$. Consider the Haar square function inequality $\|S(\sigma f)\|_{L^{p,\infty}(w)} \lesssim \|f\|_{L^p(\sigma)}$, where $\sigma(x) = w(x)^{1-p'}$ is the dual measure. Viewing this as a map from $L^p(\sigma)$ to $L^{p,\infty}(w;\ell^2)$, the dual map takes $L^{p',1}(w;\ell^2)$ to $L^{p'}(\sigma)$, and the inequality is

$$\Big\|\sum_Q |Q|^{1/2}\langle a_Q\cdot w\rangle_Q h_Q\Big\|_{L^{p'}(\sigma)} \lesssim \Big\|\Big[\sum_Q a_Q^2\Big]^{1/2}\Big\|_{L^{p',1}(w)}. \tag{3.1}$$

In the inequality, $\{a_Q\}$ are a sequence of measurable functions. We show that the implied constant is at least $C_p[w]_{A_p}^{\beta}$, for any $0 < \beta < \frac12$.

In the inequality (3.1), the right hand side is independent of the signs of the functions a_Q. Hence it, with the standard Khintchine estimate, implies the inequality

$$\Big\| \Big[\sum_Q \langle a_Q \cdot w \rangle_Q^2 \mathbf{1}_Q \Big]^{1/2} \Big\|_{L^{p'}(\sigma)} \lesssim \Big\| \Big[\sum_Q a_Q^2 \Big]^{1/2} \Big\|_{L^{p',1}(w)}$$

As the left hand side is purely positive, we prefer this form. Indeed, we specialize the inequality above to one which is of testing form. Take the functions $\{a_k\}$ to be

$$a_k(x) := c \sum_{j=k+1}^{\infty} \frac{1}{(j-k)^{\alpha}} \mathbf{1}_{[2^{-j}, 2^{-j+1})}, \qquad \tfrac{1}{2} < \alpha < 1, \ k \in \mathbb{N}.$$

For appropriate choice of constant $c = c_\alpha$, there holds $\sum_{k=1}^{\infty} a_k(x)^2 \le \mathbf{1}_{[0,1]}$, whence we have

$$\Big\| \Big[\sum_{k=1}^{\infty} a_k(x)^2 \Big]^{1/2} \Big\|_{L^{p',1}(w)} \lesssim w([0,1])^{1/p'}.$$

The next and critical point, concerns the terms $\langle a_k \cdot w \rangle_{[0,2^{-k})}$. Recalling $w(x) = |x|^{\epsilon - 1}$, we have

$$\begin{aligned} \langle a_k \cdot w \rangle_{[0,2^{-k})} &\simeq 2^k \sum_{j=k+1}^{\infty} \frac{1}{(j-k)^{\alpha}} 2^{-\epsilon j} \\ &= 2^{k(1-\epsilon)} \sum_{j=1}^{\infty} \frac{1}{j^{\alpha}} 2^{-\epsilon j} \\ &\simeq 2^{k(1-\epsilon)} \int_1^{\infty} \frac{1}{x^{\alpha}} 2^{-\epsilon x} \, dx \simeq \epsilon^{-1+\alpha} 2^{k(1-\epsilon)}. \end{aligned}$$

From this we conclude that the testing term is

$$\begin{aligned} \int_{[0,1]} \Big[\sum_{k=1}^{\infty} \langle a_k \cdot w \rangle_{[0,2^{-k})}^2 \mathbf{1}_{[0,2^{-k})} \Big]^{p'/2} d\sigma &\simeq \epsilon^{(-1+\alpha)p'} \int_{[0,1]} w(x)^{p'} \, d\sigma(x) \\ &\simeq \epsilon^{(-1+\alpha)p'} \int_{[0,1]} w(x)^{p'} w(x)^{1-p'} \, dx \\ &\simeq [w]_{A_p}^{(1-\alpha)p'} w([0,1]). \end{aligned}$$

Therefore, the power on $[w]_{A_p}$ in the implied constant in (3.1) can never be strictly less than $\frac{1}{2}$, since $\alpha > 1/2$ is arbitrary.

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